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Successive differentiation

(1) If $y = \log\left(x + \sqrt{1+x^2}\right)$, prove that $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$.

Proof:- We have $y = \log\left(x + \sqrt{1+x^2}\right)$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+\sqrt{1+x^2}}\left[1 + \frac{1}{2}(1+x^2)^{-\frac{1}{2}} \cdot 2x\right]$$

$$= \frac{1}{x+\sqrt{1+x^2}}\left[1 + \frac{x}{\sqrt{1+x^2}}\right]$$

$$\frac{dy}{dx} = \frac{1}{(x+\sqrt{1+x^2})} \cdot \frac{(\sqrt{1+x^2}+x)}{\sqrt{1+x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow (1+x^2)\left(\frac{dy}{dx}\right)^2 = 1$$

Differentiating the above equation again, we get

$$(1+x^2)2 \cdot \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} + 2x \cdot \left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow \left\{(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx}\right\} \div \frac{dy}{dx} = 0$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$$

Proved.